Addendum: Hawking Radiation of Photons in a Variable-mass Kerr Black Hole

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Hawking evaporation of photons in a variable-mass Kerr space-time is investigated by using a method of the generalized tortoise coordinate transformation. The blackbody radiant spectrum of photons displays a new spin-rotation coupling effect obviously dependent on different helicity states of photons.

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Ever since Hawking [1] initiated the discussion on particle creation by a black hole event horizon, much work has been done on the Hawking evaporation of black holes in some spherically symmetric and non-static space-times [2–4]. In the case of a non-stationary axisymmetric black hole, though the Hawking radiation of scalar particles received a fairly extensive investigation [5], similar work to that of Dirac particles encounters great obstacles. The main difficulty lies in the non-separability of the radial and angular variables for Chandrasekhar-Dirac equations [6] in the non-stationary axisymmetry space-time. In a recent paper [7] (here refer to Paper I), this dilemma has been cast off by considering simultaneously the asymptotic behaviors of the first-order and second-order forms of Dirac equation near the event horizon. A new interaction due to the coupling of the spin of Dirac particles with the rotation of the evaporating Kerr black holes was observed in the thermal radiation spectrum of Dirac particles. The character of this spin-rotation coupling effect is its obvious dependence on different helicity states of particles with spin-1/2. This effect vanishes [8] when the space-time degenerates to a spherically symmetric black hole of Vaidya-type.

In this addendum, we apply the method presented in Paper I to deal with the thermal radiation of photons in a non-stationary Kerr space-time [9,10], that is, we consider the asymptotic behaviors of the first-order and second-order forms of Maxwell equations near the event horizon. By using the relations between the first-order derivatives of three complex Newman-Penrose [11] scalars of Maxwell fields, we eliminate the crossing-terms of the first-order derivatives in the second-order equation and recast each second-order equation to a standard wave equation near the event horizon. The blackbody radiation spectrum of photons displays a spin-rotation coupling effect due to the interaction between the spin of photons and the angular momentum of radiating Kerr black holes.

As in Paper I, the line element of a variable-mass Kerr black hole [9,10] can be written in the advanced Eddington-Finkelstein system as

$$\begin{aligned} ds^2 = & \frac{\Delta - a^2\sin^2\theta}{\Sigma}dv^2 + 2\frac{r^2+a^2-\Delta}{\Sigma}a\sin^2\theta dv d\varphi - 2dvdr \\ & +2a\sin^2\theta dr d\varphi - \Sigma d\theta^2 - \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\varphi^2 \,, \end{aligned} \tag{1}$$

where $\Delta = r^2 - 2M(v)r + a^2$, $\Sigma = r^2 + a^2\cos^2\theta = \rho^*\rho$, $\rho^* = r + ia\cos\theta$, $\rho = r - ia\cos\theta$, and v is the standard advanced time. The mass M of the hole depends on the time v, but the specific angular momentum a is a constant.

The metric (1) of an evaporating Kerr black hole is a natural non-stationary generalization of the stationary Kerr solution. The geometry of this Petrov type-II space-time is characterized by three kinds of surfaces of particular interest: the apparent horizons $r_{AH}^{\pm} = M \pm (M^2 - a^2)^{1/2}$, the timelike limit surfaces $r_{TLS}^{\pm} = M \pm (M^2 - a^2\cos^2\theta)^{1/2}$, and

the event horizons $r_{EH}^{\pm} = r_H$. The event horizon is necessary a null-surface $r = r(v, \theta)$ that satisfies the null hypersurface conditions $g^{ij}\partial_i F \partial_j F = 0$ and $F(v, r, \theta) = 0$. The generalized tortoise coordinate transformation (GTCT) method is an effective one to determine the location of the event horizon of a dynamic black hole. To illuminate this method, we use it to derive the event horizon equation. Because the space-time under consideration is symmetric about φ-axis, we can introduce a GTCT [5,7] as follows

$$
\begin{aligned}
r_* &= r + \frac{1}{2\kappa(v_0, \theta_0)} \ln[r - r_H(v, \theta)]\,, \\
v_* &= v - v_0\,, \qquad\qquad \theta_* = \theta - \theta_0\,,
\end{aligned}
\tag{2}
$$

where $r_H = r(v, \theta)$ is the location of event horizon, and κ is an adjustable parameter. All parameters κ, v_0 and θ_0 characterize the initial state of the hole and are constant under the tortoise transformation.

Now applying the GTCT of Eq. (2) to the null surface equation

$$
g^{ij}\partial_i F \partial_j F = 0
$$

and then taking the $r \to r_H(v_0, \theta_0)$, $v \to v_0$ and $\theta \to \theta_0$ limits, we arrive at

$$
\Big[\Delta_H - 2(r_H^2 + a^2)\dot{r}_H + a^2 \sin^2\theta_0 \dot{r}_H^2 + {r'_H}^2\Big]\Big(\frac{\partial}{\partial r_*}F\Big)^2 = 0\,,
\tag{3}
$$

in which the vanishing of the coefficient in the square bracket can give the following equation to determine the location of the event horizon of an evaporating Kerr black hole

$$
\Delta_H - 2(r_H^2 + a^2)\dot{r}_H + a^2 \sin^2\theta_0 \dot{r}_H^2 + {r'_H}^2 = 0\,,
\tag{4}
$$

where we denote $\Delta_H = r_H^2 - 2Mr_H + a^2$. The quantities $\dot{r}_H = \partial r_H/\partial v$ is the rate of the event horizon varying in time, $r'_H = \partial r_H/\partial\theta$ is its rate changing with the angle θ. They describe the evolution of the black hole event horizon in the time and the change in the direction, which reflect the presence of quantum ergosphere near the event horizon. Eq. (4) is exactly Eq. (11) derived in Paper I.

Now we consider the sourceless Maxwell equations in the spacetime (1). When the back reaction of the massless spin-1 field on this geometry is neglected, the field equation is given by the Maxwell equations on the background spacetime (1). The sourceless Maxwell equations in the Newman-Penrose formalism [6,11] read

$$
\begin{aligned}
&(D - 2\tilde{\rho})\phi_1 - (\overline{\delta} + \tilde{\pi} - 2\alpha)\phi_0 = -\tilde{\kappa}\phi_2\,, \\
&(\delta - 2\tau)\phi_1 - (\underline{\Delta} + \mu - 2\gamma)\phi_0 = -\sigma\phi_2\,, \\
&(D + 2\epsilon - \tilde{\rho})\phi_2 - (\overline{\delta} + 2\tilde{\pi})\phi_1 = -\tilde{\lambda}\phi_0\,, \\
&(\delta + 2\beta - \tau)\phi_2 - (\underline{\Delta} + 2\mu)\phi_1 = -\tilde{\nu}\phi_0\,.
\end{aligned}
\tag{5}
$$

To write out their explicit form in the spacetime (1), we assume the complex null tetrad system established in Paper I and insert for the appropriate spin-coefficients in Eq. (A4) given in Paper I and then after substituting $\Phi_0 = \frac{\rho^*}{\sqrt{2}\rho}\phi_0$, $\Phi_1 = \rho^*\phi_1$, $\Phi_2 = \sqrt{2}\Sigma\phi_2$, into Eq. (5), we obtain

$$\begin{aligned}
\Big(\frac{\partial}{\partial r} + \frac{1}{\rho^*}\Big)\Phi_1 + \Big(\mathcal{L}_1 + \frac{ia\sin\theta}{\rho^*}\Big)\Phi_0 = 0\,,\\
\Delta\Big(\mathcal{D}_1 - \frac{1}{\rho^*}\Big)\Phi_0 - \Big(\mathcal{L}^\dagger{}_0 - \frac{ia\sin\theta}{\rho^*}\Big)\phi_1 = 0\,,\\
\Big(\frac{\partial}{\partial r} - \frac{1}{\rho^*}\Big)\Phi_2 + \Big(\mathcal{L}_0 - \frac{ia\sin\theta}{\rho^*}\Big)\Phi_1 = 0\,,\\
\Delta\Big(\mathcal{D}_0 + \frac{1}{\rho^*}\Big)\Phi_1 - \Big(\mathcal{L}^\dagger{}_1 + \frac{ia\sin\theta}{\rho^*}\Big)\Phi_2 = 2\dot{M}ria\sin\theta\Phi_0\,,
\end{aligned} \tag{6}$$

here we have defined operators

$$\begin{aligned}
\mathcal{D}_n &= \frac{\partial}{\partial r} + \frac{2}{\Delta}\Big[n(r-M) + a\frac{\partial}{\partial\varphi} + (r^2+a^2)\frac{\partial}{\partial v}\Big]\,,\\
\mathcal{L}_n &= \frac{\partial}{\partial\theta} + n\cot\theta - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} - ia\sin\theta\frac{\partial}{\partial v}\,,\\
\mathcal{L}^\dagger{}_n &= \frac{\partial}{\partial\theta} + n\cot\theta + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} + ia\sin\theta\frac{\partial}{\partial v}\,.
\end{aligned}$$

Eq. (6) can not be decoupled except in the case of a stationary Kerr black hole [6] ($M = const$). However, to deal with the problem of Hawking radiation, one should be concerned about the asymptotic behavior of Eq. (6) near the horizon only. To this end, we consider simultaneously the asymptotic behaviors of the first-order and second-order Maxwell equations near the event horizon.

First let us consider the limiting form of Eq. (6) near the event horizon. Under the transformations (2), Eq. (6) can be reduced to the following forms

$$\begin{aligned}
&\Big[\Delta_H - 2(r_H^2+a^2)\dot{r}_H\Big]\frac{\partial}{\partial r_*}\Phi_0 + \Big(r_H' + ia\sin\theta_0\dot{r}_H\Big)\frac{\partial}{\partial r_*}\Phi_1 = 0\,,\\
&\frac{\partial}{\partial r_*}\Phi_1 - \Big(r_H' - ia\sin\theta_0\dot{r}_H\Big)\frac{\partial}{\partial r_*}\Phi_0 = 0\,,\\
&\Big[\Delta_H - 2(r_H^2+a^2)\dot{r}_H\Big]\frac{\partial}{\partial r_*}\Phi_1 + \Big(r_H' + ia\sin\theta_0\dot{r}_H\Big)\frac{\partial}{\partial r_*}\Phi_2 = 0\,,\\
&\frac{\partial}{\partial r_*}\Phi_2 - \Big(r_H' - ia\sin\theta_0\dot{r}_H\Big)\frac{\partial}{\partial r_*}\Phi_1 = 0\,,
\end{aligned} \tag{7}$$

after being taken limits $r \to r_H(v_0,\theta_0)$, $v \to v_0$ and $\theta \to \theta_0$.

If the derivatives $\frac{\partial}{\partial r_*}\Phi_0$, $\frac{\partial}{\partial r_*}\Phi_1$ and $\frac{\partial}{\partial r_*}\Phi_2$ in Eq. (7) are not equal to zero, the existence condition of nontrial solutions for Φ_0, Φ_1 and Φ_2 is that the determinant of two pairs of Eq. (7) vanishes, which gives exactly the event horizon equation (4). It will be seen that the

relations (7) play a crucial role to eliminate the crossing-term of the first-order derivatives in the second-order equation.

Next we turn to the second-order form of Maxwell equations. A tedious but straightforward calculation gives

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial r}\Delta\mathcal{D}_1+\mathcal{L}^{\dagger}{}_0\mathcal{L}_1+2\rho\frac{\partial}{\partial v}\Big)\Phi_0=0\,,\\
&\Big(\frac{\partial}{\partial r}\Delta\mathcal{D}_0+\mathcal{L}^{\dagger}{}_1\mathcal{L}_0-2\rho\frac{\partial}{\partial v}+\frac{2M\rho}{\rho^{*2}}\Big)\Phi_1\\
&\qquad\equiv\Big(\Delta\mathcal{D}_1\frac{\partial}{\partial r}+\mathcal{L}_1\mathcal{L}^{\dagger}{}_0+2\rho\frac{\partial}{\partial v}+\frac{2M\rho}{\rho^{*2}}\Big)\Phi_1\\
&\qquad=2ia\sin\theta\Big[\dot{M}r(\frac{\partial}{\partial r}-\frac{1}{\rho^*})+\dot{M}\Big]\Phi_0\,,\\
&\Big(\Delta\mathcal{D}_0\frac{\partial}{\partial r}+\mathcal{L}_0\mathcal{L}^{\dagger}{}_1-2\rho\frac{\partial}{\partial v}\Big)\Phi_2\\
&\qquad=-2\ddot{M}ra^2\sin^2\theta\Phi_0-4\dot{M}ria\sin\theta\mathcal{L}_1\Phi_0\,,
\end{aligned}
\tag{8}
$$

Given the GTCT in Eq. (2), the limiting form of Eq. (8), when r approaches $r_H(v_0,\theta_0)$, v goes to v_0 and θ goes to θ_0, reads

$$
\begin{aligned}
&\Big\{\Big[\frac{r_H(1-2\dot{r}_H)-M}{\kappa}+2\Delta_H-2\dot{r}_H(r_H^2+a^2)\Big]\frac{\partial^2}{\partial r_*^2}-2r'_H\frac{\partial^2}{\partial r_*\partial\theta_*}\\
&+2a(1-\dot{r}_H)\frac{\partial^2}{\partial r_*\partial\varphi}+2(r_H^2+a^2-\dot{r}_Ha^2\sin^2\theta_0)\frac{\partial^2}{\partial r_*\partial v_*}-\Big[2(M-r_H)\\
&+2(r_H-ia\cos\theta_0)\dot{r}_H+r'_H\cot\theta_0+r''_H+\ddot{r}_Ha^2\sin^2\theta_0\Big]\frac{\partial}{\partial r_*}\Big\}\Phi_0=0\,,
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\Big\{\Big[\frac{r_H(1-2\dot{r}_H)-M}{\kappa}+2\Delta_H-2\dot{r}_H(r_H^2+a^2)\Big]\frac{\partial^2}{\partial r_*^2}-2r'_H\frac{\partial^2}{\partial r_*\partial\theta_*}\\
&+2a(1-\dot{r}_H)\frac{\partial^2}{\partial r_*\partial\varphi}+2(r_H^2+a^2-\dot{r}_Ha^2\sin^2\theta_0)\frac{\partial^2}{\partial r_*\partial v_*}\\
&-\Big(-2r_H\dot{r}_H+r'_H\cot\theta_0+r''_H+\ddot{r}_Ha^2\sin^2\theta_0\Big)\frac{\partial}{\partial r_*}\Big\}\Phi_1\\
&=-2\dot{M}r_Hia\sin\theta_0\frac{r'_H+ia\sin\theta_0\dot{r}_H}{\Delta_H-2(r_H^2+a^2)\dot{r}_H}\frac{\partial}{\partial r_*}\Phi_1\,,
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
&\Big\{\Big[\frac{r_H(1-2\dot{r}_H)-M}{\kappa}+2\Delta_H-2\dot{r}_H(r_H^2+a^2)\Big]\frac{\partial^2}{\partial r_*^2}-2r'_H\frac{\partial^2}{\partial r_*\partial\theta_*}\\
&+2a(1-\dot{r}_H)\frac{\partial^2}{\partial r_*\partial\varphi}+2(r_H^2+a^2-\dot{r}_Ha^2\sin^2\theta_0)\frac{\partial^2}{\partial r_*\partial v_*}-\Big[2(r_H-M)
\end{aligned}
$$

$$+2(-3r_H+ia\cos\theta_0)\dot r_H+r'_H\cot\theta_0+r''_H+\ddot r_H a^2\sin^2\theta_0\Big]\frac{\partial}{\partial r_*}\Big\}\Phi_2$$

$$=-4\dot M r_H ia\sin\theta_0\frac{r'_H+ia\sin\theta_0\dot r_H}{\Delta_H-2(r_H^2+a^2)\dot r_H}\frac{\partial}{\partial r_*}\Phi_2\,, \tag{11}$$

where we have replaced the first-order derivative term $\frac{\partial}{\partial r_*}\Phi_0$ in Eqs. (10, 11) by using the relations (7). In the calculations, the L'Hôsptial's rule has been used to treat an infinite form of 0/0-type.

In order to reduce Eqs. (9), (10) and (11) to a standard form of wave equation near the event horizon, we adjust the parameter κ as did in Paper I, then these wave equations can be recast into an united form as follows

$$\Big[\frac{\partial^2}{\partial r_*^2}+2\frac{\partial^2}{\partial r_*\partial v_*}+2\Omega\frac{\partial^2}{\partial r_*\partial\varphi}+2C_3\frac{\partial^2}{\partial r_*\partial\theta_*}+2(C_2+iC_1)\frac{\partial}{\partial r_*}\Big]\Psi_p=0\,, \tag{12}$$

where the angular velocity of the event horizon of the evaporating Kerr black hole, Ω, and the coefficient C_3 are presented in Paper I, while both C_1 and C_2 are real,

$$\begin{aligned}C_2+iC_1=&\frac{-1}{2(r_H^2+a^2-\dot r_H a^2\sin^2\theta_0)}\Big[2p(r_H-M+ia\cos\theta_0\dot r_H)\\&-2(2p+1)r_H\dot r_H+r'_H\cot\theta_0+r''_H+\ddot r_H a^2\sin^2\theta_0\\&+2(s+p)\dot M r_H ia\sin\theta_0\frac{r'_H+ia\sin\theta_0\dot r_H}{\Delta_H-2\dot r_H(r_H^2+a^2)}\Big]\,,\end{aligned}$$

in which the correspondence should be $\Psi_p=\Phi_0,\ \Phi_1,\ \Phi_2$ when $p=-1,0,1$ for ($s=1$), respectively. It should be pointed out that Eq. (12) includes, as a special case, the wave equation (19) of Dirac particles discussed in Paper I, that is, $\Psi_p=P_2,\ P_1$, when $p=-1/2,1/2$ for ($s=1/2$).

The subsequent analysis parallels the treatment of Paper I, and one can obtain the Hawking radiation spectrum of photons from the black hole by using the method of Damour-Ruffini-Sannan's [12],

$$\langle\mathcal{N}_\omega\rangle\sim\frac{1}{e^{(\omega-m\Omega-C_1)/T}-1}\,,\qquad T=\frac{\kappa}{2\pi}\,. \tag{13}$$

where ω is the energy of photons, m is the azimuthal quantum number. The explicit expression of the surface gravity κ is referred to Paper I.

The thermal radiation spectrum (13) due to the Bose-Einstein statistics of photons shows that the black hole emits radiation just like a black body. It demonstrate that the energy spectrum of photons in an evaporating Kerr space-time is composed of two parts:

$$\begin{aligned}\omega_p=&\frac{a}{r_H^2+a^2-\dot r_H a^2\sin^2\theta_0}\Big[m(1-\dot r_H)-p\cos\theta_0\dot r_H\\&+(s+p)\dot M r_H\frac{\sin\theta_0 r'_H}{\Delta_H-2\dot r_H(r_H^2+a^2)}\Big]\,,\end{aligned} \tag{14}$$

the rotational energy $m\Omega$ arising from the coupling of the orbital angular momentum of photons with the rotation of the black hole and C_1 due to the coupling of the intrinsic spin of photons and the angular momentum of the black hole. From the explicit expression of the "spin-dependent" term C_1

$$C_1 = \frac{\Omega}{1-\dot{r}_H}\Big[-p\cos\theta_0\dot{r}_H + (s+p)\dot{M}r_H\frac{\sin\theta_0 r'_H}{\Delta_H - 2\dot{r}_H(r_H^2+a^2)}\Big]\,, \tag{15}$$

one can easily find that it vanishes in the case of a stationary Kerr black hole ($M = const$, $\dot{r}_H = r'_H = 0$) or a Vaidya-type black hole ($a = 0$, $r'_H = 0$, $\dot{r}_H \neq 0$). The term C_1 is obviously related to the helicity of photons in different spin states, it characterizes a new effect arising from the interaction between the spin of photons and the rotation of an evaporating black hole.

In this addendum, we have dealt with Hawking radiation of photons in a variable-mass Kerr black hole. Eq. (13) shows the thermal radiation spectrum of photons in the non-stationary Kerr space-time, in which an extra term C_1 represents a new spin-rotation coupling effect probably arising from the interaction between the spin of photons and the angular momentum of the evaporating Kerr black hole. The feature of this spin-rotation coupling effect is its dependence on different helicity states of photons. As is pointed out in Paper I, this effect vanishes when a black hole is stationary ($\dot{r}_H = r'_H = 0$) or it has a zero angular momentum ($a = 0$).

In summary, this study confirms that the thermal radiation spectra of particles with higher spins in the non-stationary Kerr black hole displays a new effect due to the coupling of the intrinsic spin of particles with the rotation of the black holes. The spin-rotation coupling effect shows that a non-stationary Kerr space-time has some distinct effects different from that of a stationary Kerr black hole. This effect is not shared by a stationary Kerr black hole or a Vaidya-type spherically symmetric black hole.

This work is supported in part by the NSFC in China.

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[1] Hawking, S. W. (1974). *Nature*, **248**, 30; (1975). *Commun. Math. Phys.* **43**, 199.

[2] Kim, S. W., Choi, E. Y., Kim, S. K., and Yang, Y. (1989). *Phys. Lett. A* **141**, 238.

[3] Zhao, Z., Yang, C. Q., and Ren, Q. A. (1992). *Gen. Rel. Grav.* **26**, 1055; Li, Z. H., and Zhao, Z. (1993). *Chin. Phys. Lett.* **10**, 126; Zhu, J. Y., Zhang, J. H., and Zhao, Z. (1994). *Int. J.*

Theor. Phys. **33**, 2137; (1994). *Science in China A* **24**, 1056; Yang, B., and Zhao, Z. (1994). *Acta Physica Sinica*, **43**, 858.

[4] Ma, Y., and Yang, S. Z. (1993). *Int. J. Theor. Phys.* **32** (1993) 1237; Zhang, J. H., and Guo, S. Y. (1995). *Acta Mathematica Scientia*, **15**, 422; Zhang, J. H., Meng, Q. M., and Li, C. A. (1996). *Acta Physica Sinica*, **45**, 177; Ma, Y., and Yang, S. Z. (1997). *Acta Physica Sinica*, **46**, 2280.

[5] Zhao, Z., Dai, X. X., and Huang, W. H. (1993). *Acta Astrophysica Sinica*, **13**, 299; Luo, M. W. (2000). *Acta Physica Sinica*, **49**, 1035; Jing, J. L., and Wang, Y. J. (1997). *Int. J. Theor. Phys.* **36**, 1745.

[6] Chandrasekhar, S. (1983). *The Mathematical Theory of Black Holes*, (Oxford University Press, New York).

[7] Wu, S. Q., and Cai, X. (2001). *Chin. Phys. Lett.* **18**, 485; (2001). *Gen. Rel. Grav.* **33**(7), (in press).

[8] Wu, S. Q. and Cai, X. (2001). *Int. J. Theor. Phys.* **40**, 1349; (2001). Mod. Phys. Lett. A **16**, 1549.

[9] Gonzalez, C., Herrera, L., and Jimenez, J., (1979). *J. Math. Phys.* **20**, 837; Jing, J. L., and Wang, Y. J. (1996). *Int. J. Theor. Phys.* **35**, 1481.

[10] Carmeli, M., and Kaye, M. (1977). *Ann. Phys. (NY)* **103**, 197; Carmeli, M. (1982). *Classical Fields: General Relativity and Gauge Theory*, (John Wiley & Sons, New York).

[11] Newman, E., and Penrose, R. (1962). *J. Math. Phys.* **3**, 566.

[12] Damour, T., and Ruffini, R. (1976). *Phys. Rev. D* **14**, 332; Sannan, S. (1988). *Gen. Rel. Grav.* **20**, 239.